Complex Nos. - Supplementary Notes

Sept. 26, 2025

Jake Bobowski

Recall: Any complex no. can be expressed in two ways.

Component form: $z = x + jy$

Magnitude & Phase: $z = |z| e^{j\phi}$

$= |z| (\cos\phi + j\sin\phi)$

Converting between the two representations:

$$x = Re[z] = |z| \cos\phi$$

$$y = IM[z] = |z| \sin\phi$$

$$|z| = \sqrt{x^2 + y^2}$$

$$\tan\phi = \frac{y}{x}$$

Recall the analogy w/ 2-D vectors

$$\vec{V} = x\hat{i} + y\hat{j}$$

We can find the length of $\vec{V}$ by taking the dot product of $\vec{V}$ w/ itself.

$$|\vec{V}|^2 = \vec{V} \cdot \vec{V}$$
$$= (x\hat{i} + y\hat{j}) \cdot (x\hat{i} + y\hat{j})$$
$$= x^2 + y^2 \quad \left[\text{using } \hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = 1 \text{ and } \hat{i} \cdot \hat{j} = 0 \right]$$

The length or magnitude of a complex no. is found by multiplying z by its complex conjugate.

If $z = x + jy$, then the definition of the complex conjugate of z is:

$$z^* = x - jy$$

We take z and replace j w/ $-j$

So, if $z = |z| e^{j\phi}$, then $z^* = |z| e^{-j\phi}$

Now consider $zz^* = (x + jy)(x - jy)$

$= x^2 - \cancel{jxy} + \cancel{jxy} + y^2$

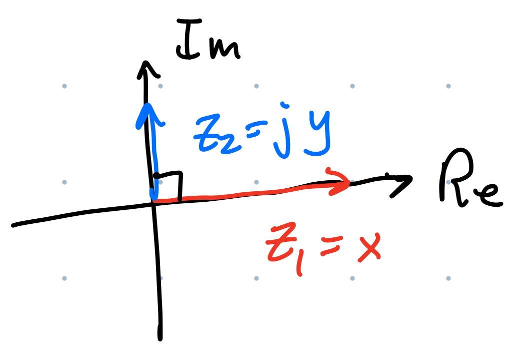
$= x^2 + y^2 = |z|^2$.

or $zz^* = \left(|z| e^{j\phi}\right)\left(|z| e^{-j\phi}\right)$

$= |z|^2 \cancel{e^{j\phi}} \cancel{e^{-j\phi}}$

$\therefore |z|^2 = zz^*$ gives us the square of the magnitude of z.

Consider two $\perp$ complex nos.



$$z_1 z_2^* = x(-jy) = -jxy \neq 0$$

$\therefore$ zz^* is not a perfect analogy to the dot product since, if $\vec{V}_1 \perp \vec{V}_2$, then

$$\vec{V}_1 \cdot \vec{V}_2 = 0.$$

However, as we'll attempt to show $z_1^* z_2$ is actually more general than a dot product.

Consider two aribrary complex numbers:

$$z_1 = x_1 + jy_1 \qquad z_2 = x_2 + jy_2$$

$$z_1^* = x_1 - jy_1$$

then
$$z_1^* z_2 = (x_1 - jy_1)(x_2 + jy_2)$$
$$= x_1 x_2 - \underbrace{j^2}_{+1} y_1 y_2 + j x_1 y_2 - j y_1 x_2$$

$$\therefore z_1^* z_2 = \underbrace{(x_1 x_2 + y_1 y_2)}_{\text{like } \vec{V}_1 \cdot \vec{V}_2} + j \underbrace{(x_1 y_2 - y_1 x_2)}_{\text{like } \hat{k} \text{ component of } \vec{V}_1 \times \vec{V}_2} \text{ ⑪}$$

same

$$\vec{V}_1 \cdot \vec{V}_2 = (x_1 \hat{i} + y_1 \hat{j}) \cdot (x_2 \hat{i} + y_2 \hat{j}) = x_1 x_2 + y_1 y_2$$

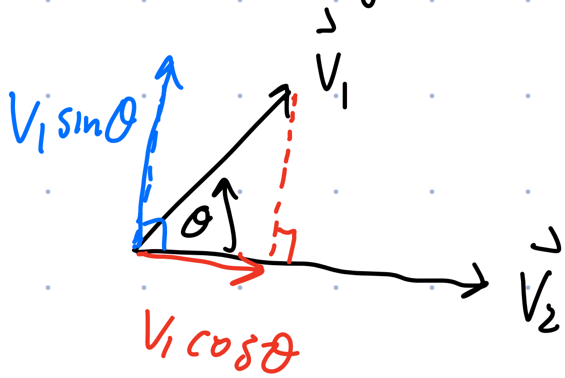
$$\vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \end{vmatrix} = \hat{k}\,(x_1 y_2 - y_1 x_2)$$

same

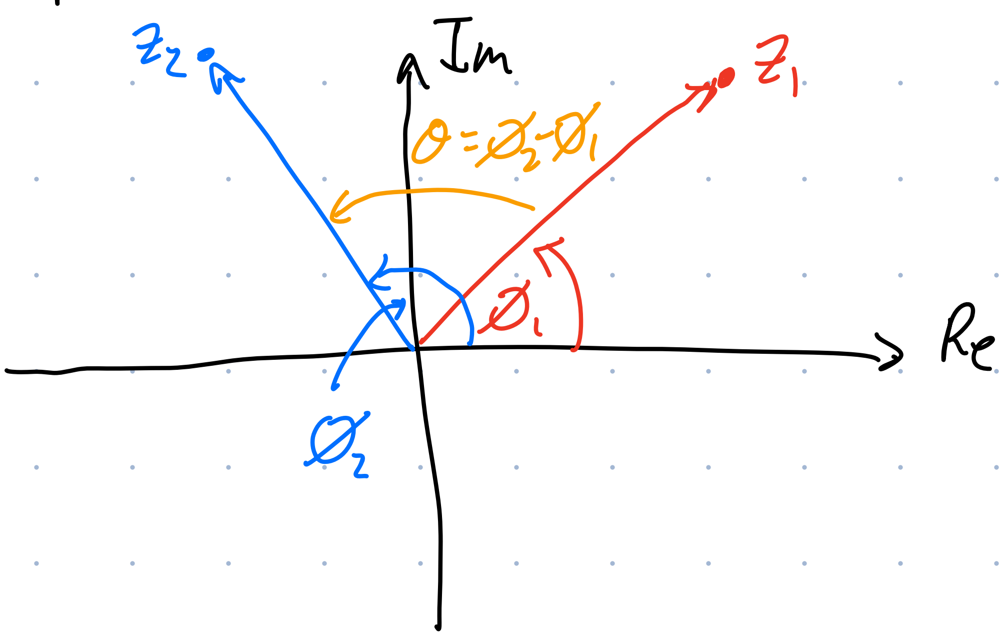
But we already know from vector algebra that

$$\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2 \cos\theta$$

$$\vec{V}_1 \times \vec{V}_2 = V_1 V_2 \sin\theta$$



For complex numbers in the complex plane:



$$\therefore \quad z_1^* z_2 = |z_1||z_2|\cos(\phi_2-\phi_1) + j|z_1||z_2|\sin(\phi_2-\phi_1)$$

$$\text{If} \quad \underbrace{\phi_2-\phi_1 = 0}_{z_1 \parallel z_2} \quad \Rightarrow \quad z_1^* z_2 = |z_1||z_2|$$

$$\text{If} \quad \underbrace{\phi_2-\phi_1 = \pm\pi}_{z_1 \text{ antiparallel to } z_2} \quad \Rightarrow \quad z_1^* z_2 = -|z_1||z_2|$$

$$\boxed{\mathrm{Im}[z_1^* z_2] = 0 \Rightarrow \begin{cases} z_1 \parallel z_2 \text{ if } \mathrm{Re}[z_1^* z_2] > 0 \\ z_1 \text{ antiparallel } z_2 \text{ if } \mathrm{Re}[z_1^* z_2] < 0 \end{cases}}$$

$$\text{If} \quad \underbrace{\phi_2-\phi_1 = \pm\frac{\pi}{2}}_{z_1 \perp z_2} \qquad z_1^* z_2 = \pm j|z_1||z_2|$$

$$\boxed{\mathrm{Re}[z_1^* z_2] = 0 \Rightarrow z_1 \perp z_2}$$

Of course, we could have also obtained these results from

$$z_1 = |z_1| e^{j\phi_1} \qquad z_2 = |z_2| e^{j\phi_2}$$

$$z_1^* = |z_1| e^{-j\phi_1}$$

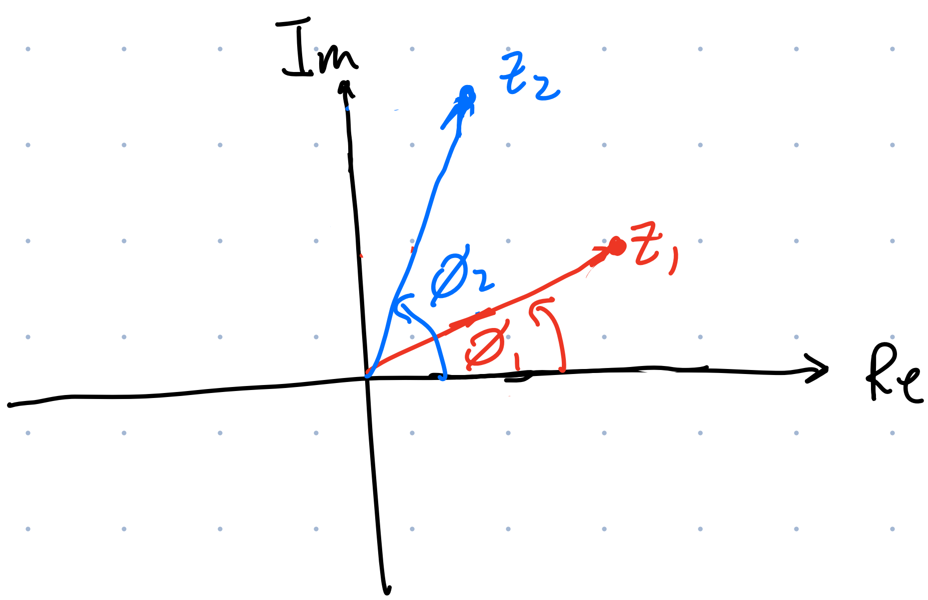
$$\therefore \; z_1^* z_2 = |z_1||z_2| e^{j\phi_2} e^{-j\phi_1}$$

$$= |z_1||z_2| e^{j(\phi_2 - \phi_1)}$$

$$= |z_1||z_2| \left[\cos(\phi_2 - \phi_1) + j \sin(\phi_2 - \phi_1) \right]$$

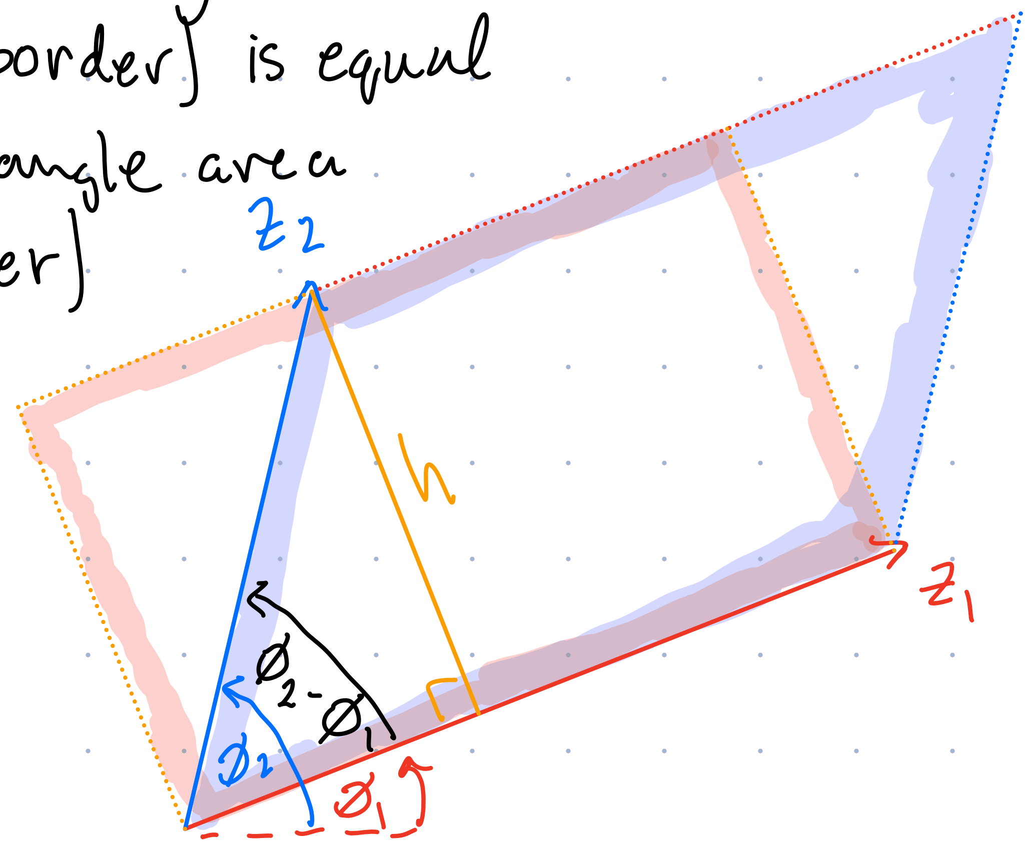
same as # !

Interpret $\mathrm{Im}[z_1^* z_2] = |z_1||z_2| \sin(\phi_2 - \phi_1)$



Can form a parallelogram from z_1 & z_2

The parallelogram area (blue border) is equal to rectangle area (red border)



$$\therefore \underbrace{|z_1| h}_{\text{rectangle area}} = A_{para}.$$

$$\therefore A_{para} = |z_1| |z_2| \sin(\phi_2 - \phi_1)$$

$\therefore \text{Im}[z_1^* z_2]$ calculates the area of the parallelogram formed by z_1 & z_2.

Consider adding two complex nos.

$$z_1 = x_1 + jy_1 = |z_1| e^{j\phi_1}$$
$$z_2 = x_2 + jy_2 = |z_2| e^{j\phi_2}$$

Adding in magnitude & phase form is not so convenient

$$z_1 + z_2 = |z_1| e^{j\phi_1} + |z_2| e^{j\phi_2}$$

No simplification possible. Just like I wouldn't add vectors using magnitude and angle.

Add the component forms is much more natural.

$$z_{net} = z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$
$$= (x_1 + x_2) + j(y_1 + y_2)$$

Now, if we want, we can convert to the other form:

$$z_{net} = |z_{net}| e^{j\phi_{net}}$$

where $|z_{net}|^2 = (x_1 + x_2)^2 + (y_1 + y_2)^2$

$$\tan \phi_{net} = \frac{y_1 + y_2}{x_1 + x_2}$$

What about products of complex nos.?

Component form is possible:

$$z_3 = z_1 z_2 = (x_1 + jy_1)(x_2 + jy_2)$$
$$= x_1 x_2 + j x_1 y_2 + j x_2 y_1 + \underbrace{j^2}_{-1} y_1 y_2$$

$$\therefore z_3 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1)$$

or $z_3 = |z_3| e^{j\phi_3}$

$$|z_3|^2 = (x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2$$

Let's expand and see what happens...

$$x_1^2 x_2^2 + y_1^2 y_2^2 - \cancel{2 x_1 x_2 y_1 y_2}$$

$$+ x_1^2 y_2^2 + x_2^2 y_1^2 + \cancel{2 x_1 x_2 y_1 y_2}$$

$$= x_1^2 \underbrace{(x_2^2 + y_2^2)}_{|z_2|^2} + y_1^2 \underbrace{(x_2^2 + y_2^2)}_{|z_2|^2}$$

$$= \underbrace{(x_1^2 + y_1^2)}_{|z_1|^2} |z_2|^2$$

$$\therefore |z_3|^2 = |z_1|^2 |z_2|^2 \Longrightarrow \boxed{|z_3| = |z_1||z_2|}$$

The magnitudes multiply, perhaps, as expected.

What about the phase?

$$\tan \phi_3 = \frac{\mathrm{Im}[z_3]}{\mathrm{Re}[z_3]} = \frac{x_1 y_2 + x_2 y_1}{x_1 x_2 - y_1 y_2}$$

divide by $x_1 x_2$

Let's employ a lesser-known identity:

$$\tan^{-1}\left(\frac{u+v}{1-uv}\right) = \tan^{-1} u + \tan^{-1} v$$

for $uv > 1$

$$\tan \phi_3 = \frac{\frac{y_2}{x_2} + \frac{y_1}{x_1}}{1 - \frac{y_1}{x_1}\frac{y_2}{x_2}}$$

take $u = \frac{x_1}{y_1}$, $v = \frac{x_2}{y_2}$

$\therefore uv > 1$ if $x_1, x_2, y_1, y_2 > 0$

$$\therefore \phi_3 = \underbrace{\tan^{-1}\left(\frac{y_1}{x_1}\right)}_{\phi_1} + \underbrace{\tan^{-1}\left(\frac{y_2}{x_2}\right)}_{\phi_2}$$

$$\boxed{\therefore \phi_3 = \phi_1 + \phi_2}$$

When multiplying complex nos, the magnitudes multiply and the phases add. Of course, we could have arrived at this conclusion much more easily by using the magnitude and phase form:

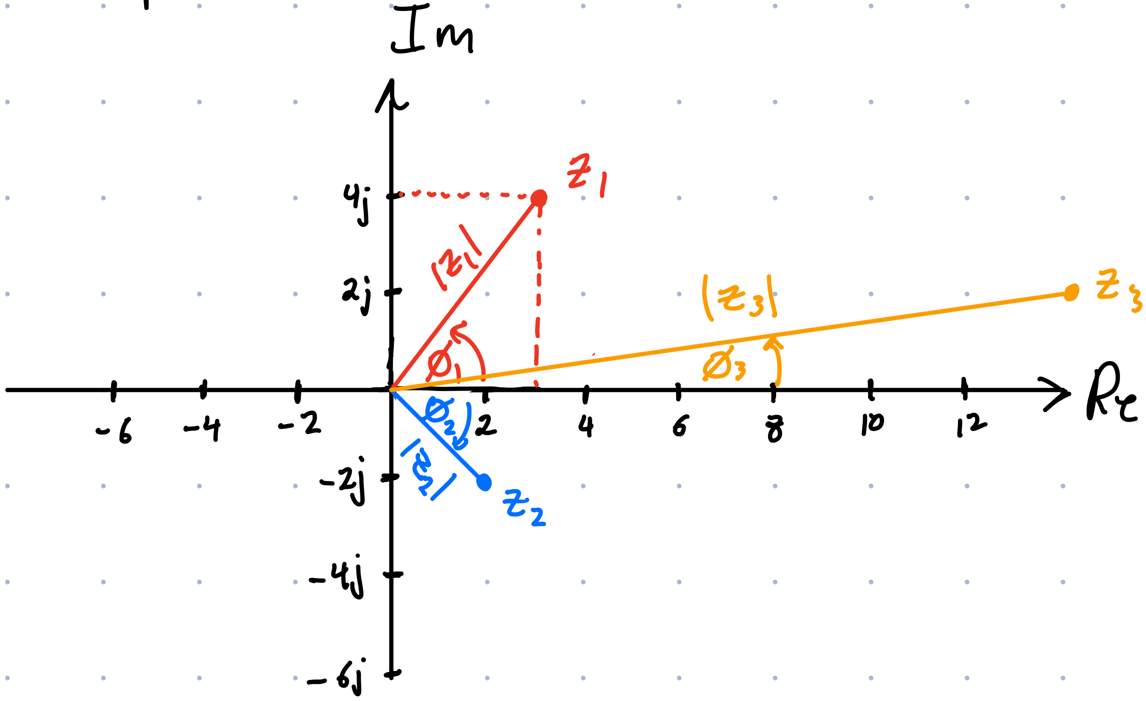
$$z_3 = z_1 z_2$$

$$= \left(|z_1| e^{j\phi_1}\right)\left(|z_2| e^{j\phi_2}\right)$$

$$\underline{|z_3|} e^{j\underline{\phi_3}} = \underline{|z_1||z_2|} e^{j(\underline{\phi_1+\phi_2})}$$

$$\left.\begin{aligned} \therefore |z_3| &= |z_1||z_2| \\ \phi_3 &= \phi_1 + \phi_2 \end{aligned}\right\} \text{ same as before.}$$

Graphical picture:



$$z_1 = 3+4j = 5e^{j\phi_1} \qquad \tan\phi_1 = \frac{4}{3}$$

$$\Rightarrow \phi_1 = 53.1^\circ$$

$$z_2 = 2-2j = 2\sqrt{2}e^{j\phi_2} \qquad \tan\phi_2 = -1$$

$$\Rightarrow \phi_2 = -45^\circ$$

$$\therefore \phi_3 = \phi_1 + \phi_2 = 53.1^\circ - 45^\circ$$

$$= 8.1^\circ$$

$$|z_3| = |z_1||z_2| = 10\sqrt{2} = 14.1$$

$$\therefore x_3 = |z_3|\cos\phi_3 = 13.9$$

$$y_3 = |z_3|\sin\phi_3 = 2.0$$

Division

$$z_3 = \frac{z_2}{z_1} = \frac{|z_2|e^{j\phi_2}}{|z_1|e^{j\phi_1}} = \frac{|z_2|}{|z_1|}e^{j\phi_2}e^{-j\phi_1}$$

$$= \frac{|z_2|}{|z_1|}e^{j(\phi_2-\phi_1)}$$

$$\boxed{\therefore |z_3| = \frac{|z_2|}{|z_1|} \qquad \phi_3 = \phi_2 - \phi_1}$$

Return to Euler's equation:

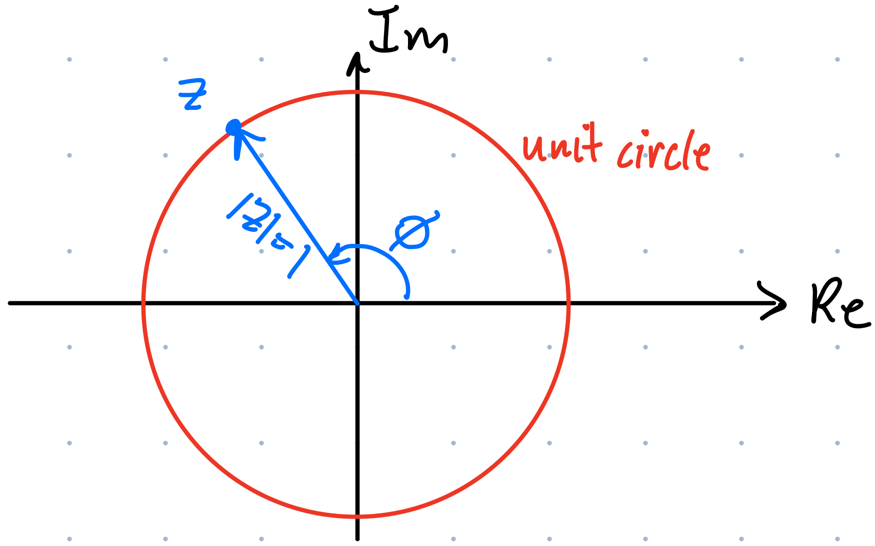
$$z = e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

Consider $|z|^2 = zz^* = e^{\pm j\phi} e^{\mp j\phi} = 1$

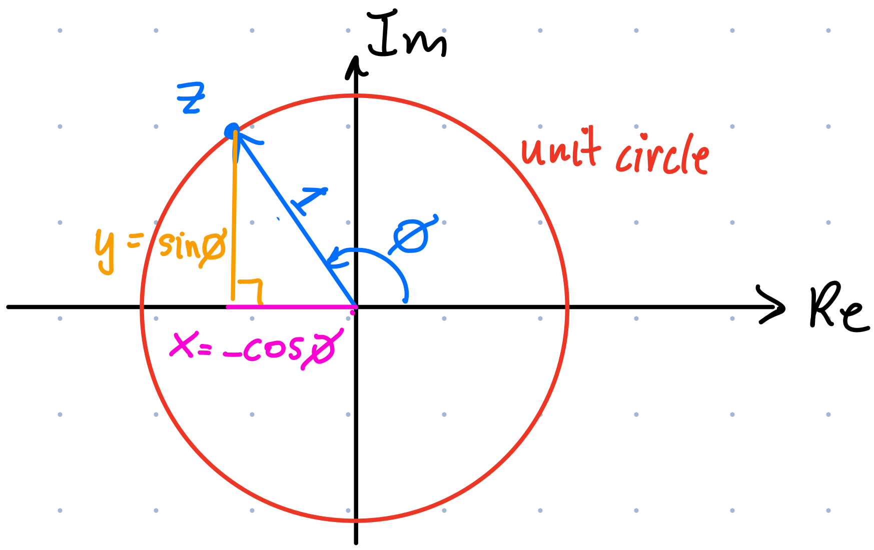
$\therefore |z| = 1$.

The length or magnitude of $z = e^{\pm j\phi}$ is 1. $\therefore$ The distance from the origin of the complex plane to any pt. represented by $z = e^{j\phi}$ is one.

In other words, it is convenient to think of $e^{j\phi}$ as mapping to the set of all points on the unit circle in the complex plane.



Note that the unit circle and $x = \cos\phi$ & $y = \sin\phi$ create right angle triangles which explains why trig. is associated w/ triangles.



However, we should also recognize that $e^{j\phi} = \cos\phi + j\sin\phi$ is also connected to circular motion and, in the case of uniform circular motion, simple harmonic motion or oscillations.

See the next couple of pages.

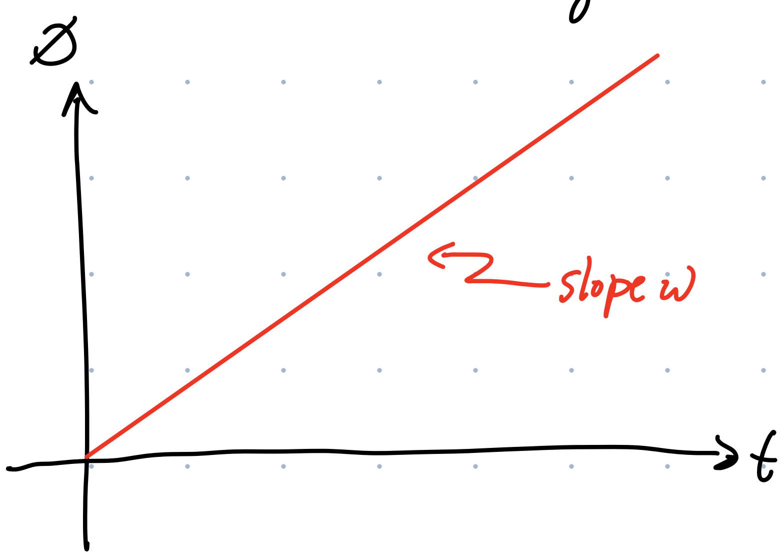
Why are complex exponentials so prevelent in physics?

→ They are very effective at describing waves. ⇒ E&M, QM, optics, electronics...

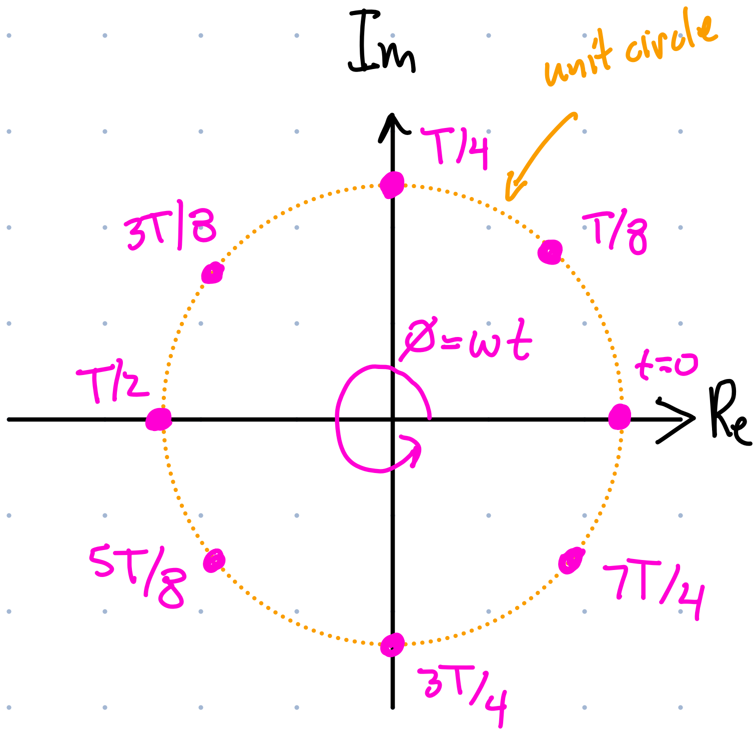
Even when not describing pure oscillitory phenomena, as you move through the physics program you will think more and more in terms of frequency. Leads to Fourier transforms which decompose functions of time/position into expansions of harmonic fcns (frequency/wavevector domain). $e^{j\omega t}$ / e^{jkx} will play essential roles.

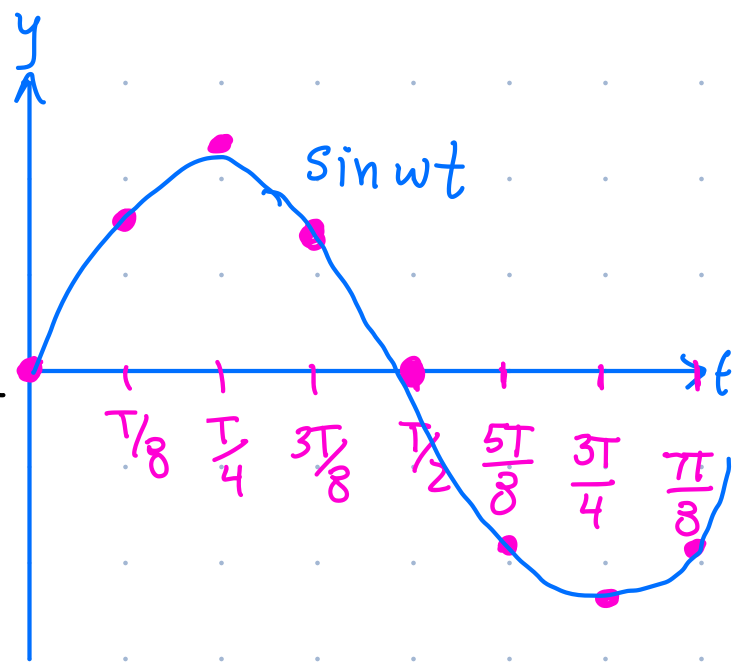
Consider $z = e^{j\omega t} = \cos\omega t + j\sin\omega t$

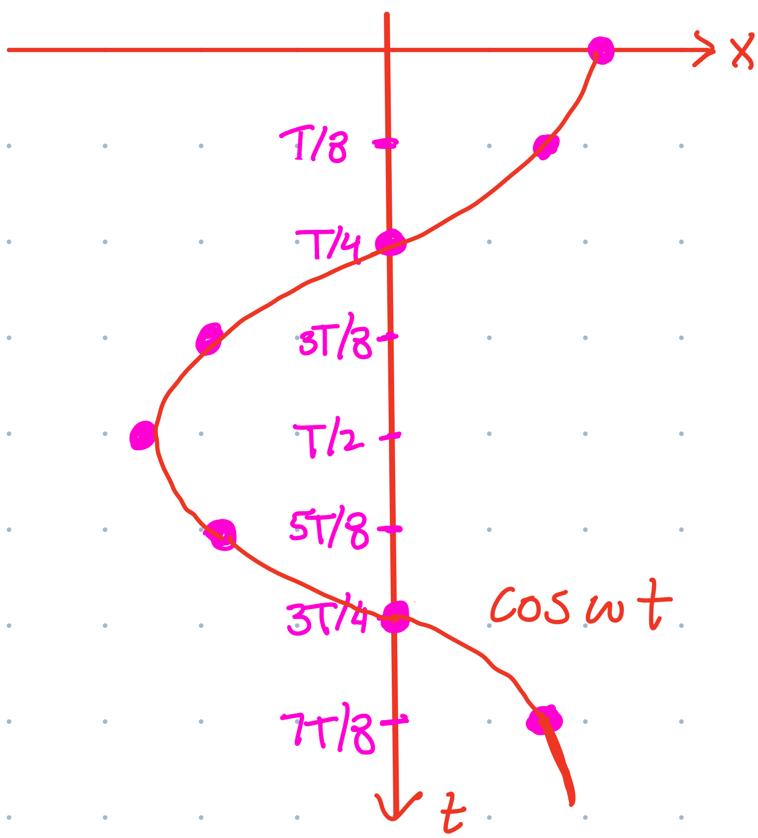
Here, the phase $\phi = \omega t$ increases linearly with time.



A linearly increasing ϕ corresponds to uniform circular motion in the complex plane. As we'll see, this uniform circular motion encodes the sinusoidal oscillation.



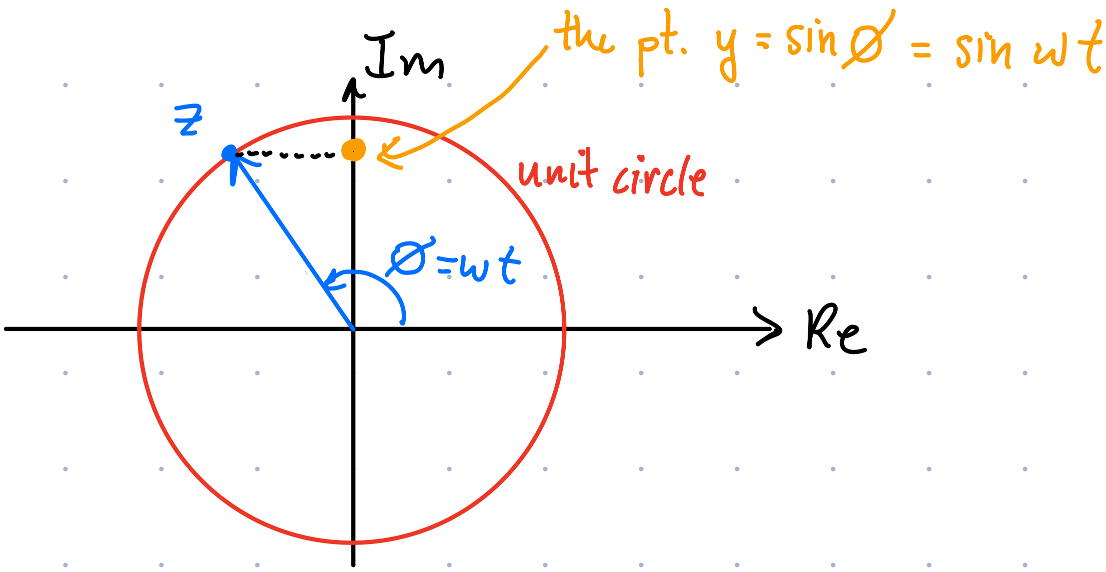




∴ we can see that uniform circular motion in the complex plane naturally encodes $\cos \omega t$ & $\sin \omega t$ which helps to, to some extent, demystify Euler's Eq'n

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi$$

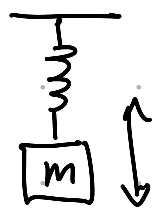
Consider the projection of $z = e^{j\omega t}$ onto the y-axis



If we observe the orange point on the y-axis as a fcn of time, it will undergo simple harmonic motion (SHM) w/ angular freq. $\omega = 2\pi f = \frac{2\pi}{T}$

where f is the freq. and T is the period (time to complete one full cycle).

A mass on a spring undergos SHM. Its position is a solution to the equation of motion



$$F = ma = -ky$$

or $$m\frac{d^2y}{dt^2} = -ky$$

or $$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$ It is common to define $\omega^2 = \frac{k}{m}$

$$\therefore \frac{d^2y}{dt^2} = -\omega^2$$

amplitude of osc.

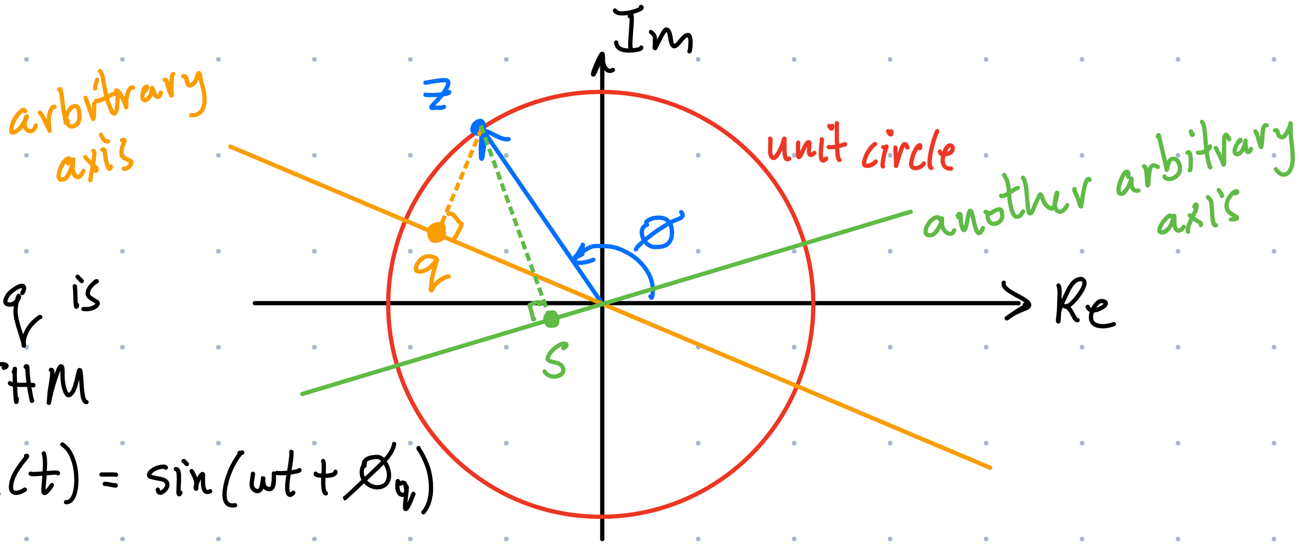
The solution for y is $y = A\sin(\omega t + \phi_0)$

phase determined by initial conditions

A & ϕ_0 are the integration constants left over from solving the 2nd order differential equation.

$\therefore$ The point y along the imaginary axis moves just like a mass on a spring. It is fastest through the origin and has turning points at $y = \pm 1$.

In fact, the motion of a point along any axis that passes through the origin of the complex plane undergos SHM.



Pt. q is in SHM

$$q(t) = \sin(\omega t + \varnothing_q)$$

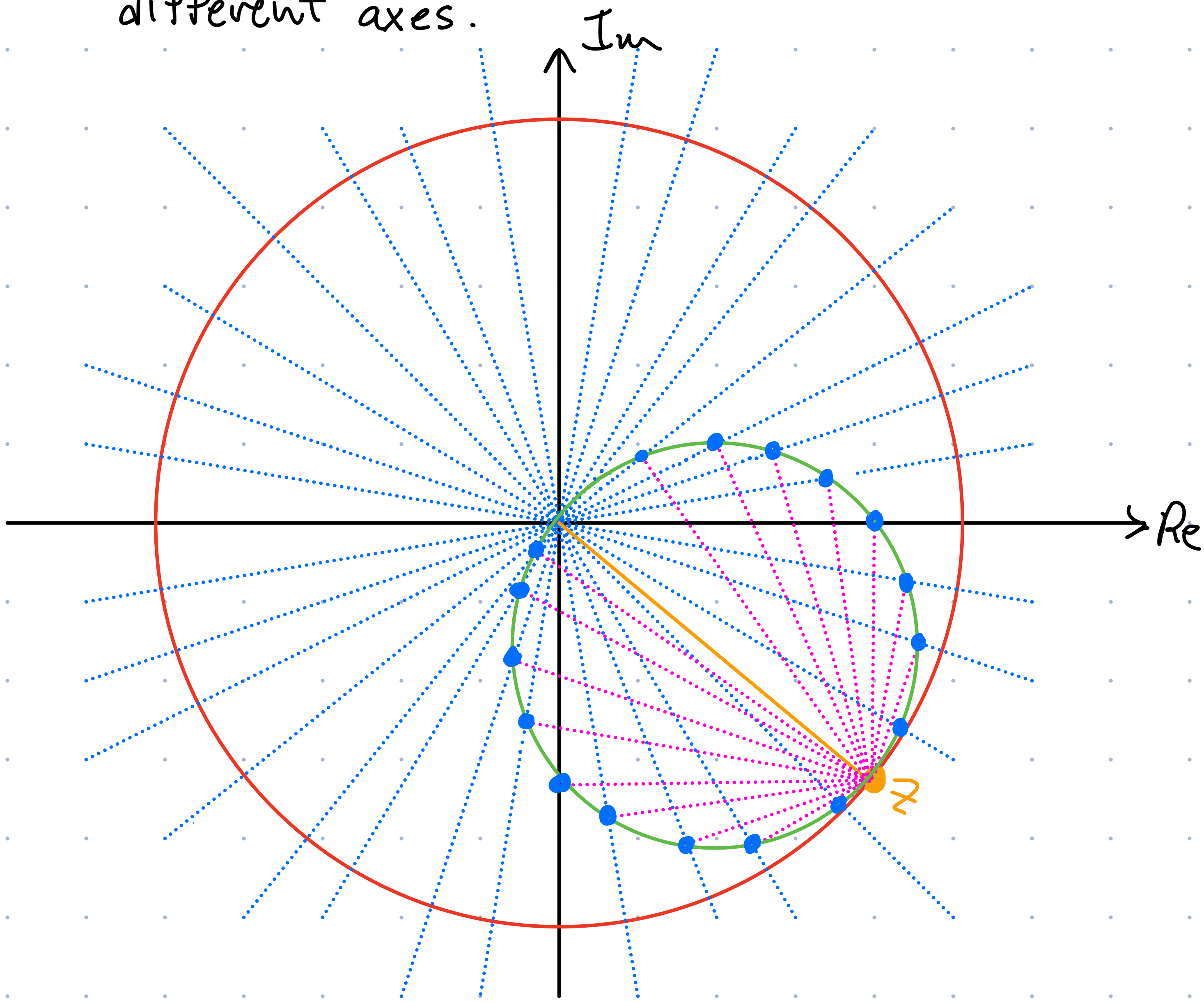
Pt. s is in SHM

$$s(t) = \sin(\omega t + \varnothing_s)$$

q, s oscillate w/ the same freq $\omega = 2\pi f$, but have different phases $\varnothing_q$, $\varnothing_s$.

The phases work out such that no two points ever cross/meet. That is, for example, no two points ever pass through the origin at the same time.

Just for fun, let's see what happens when we have many points osc. along many different axes.



The projections of $z = e^{j\phi}$ onto arbitrary axes through the origin are all undergoing SHM if $\phi = \omega t$. None of the points ever meet and the form a circle of radius $1/2$ that fits inside the unit circle.

The inner circle always passes through the origin and one point on the unit circle.

If $\phi = \omega t$, the inner circle will rotate/orbit around the origin w/ period $T = \frac{2\pi}{\omega}$.

I saw this construction in a great Numberphile video. I would recommend it to anyone w/ any interest in Math. Here's the url:

https://youtu.be/snHKEpCv0Hk?si=GltiT-tKOYYZJDux

Alternatively, just search for "Numberphile trig".

consider $e^{j\phi} + e^{-j\phi} = [\cos\phi + j\sin\phi] + [\cos\phi - j\sin\phi] = 2\cos\phi$

$e^{j\phi} - e^{-j\phi} = [\cos\phi + j\sin\phi] - [\cos\phi - j\sin\phi] = 2j\sin\phi$

$$\Downarrow$$

$$\boxed{\begin{aligned} &\text{a} \quad \cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \\ &\text{b} \quad \sin\phi = \frac{e^{j\phi} - e^{-j\phi}}{2j} \\ &\text{c} \quad \tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{1}{j}\left[\frac{e^{j\phi} - e^{-j\phi}}{e^{j\phi} + e^{-j\phi}}\right] \end{aligned}}$$

These relations can be used to derive trig identities in a straightforward way.

Consider $a^2 + b^2$

$$\cos^2\phi + \sin^2\phi = \left(\frac{e^{j\phi}+e^{-j\phi}}{2}\right)^2 + \left(\frac{e^{j\phi}-e^{-j\phi}}{2j}\right)^2$$

$$= \frac{1}{4}\left[\left(\cancel{e^{2j\phi}} + \cancel{e^{-2j\phi}} + 2\right) - \left(\cancel{e^{2j\phi}} + \cancel{e^{-2j\phi}} - 2\right)\right]$$

$$= \frac{1}{4}[4] = 1 \checkmark$$

$$\boxed{\therefore \cos^2\phi + \sin^2\phi = 1}$$

Consider $\left(e^{\pm j\phi}\right)^2 = \left(\cos\phi \pm j\sin\phi\right)^2$

$$\therefore \underbrace{e^{\pm 2j\phi}}_{\cos 2\phi \pm j\sin 2\phi} = \cos^2\phi + j^2\sin^2\phi \pm 2j\cos\phi\sin\phi$$

$$= \underline{\cos^2\phi - \sin^2\phi} \pm \underline{2}j\,\underline{\cos\phi\sin\phi}$$

Equate real and imaginary parts...

$$\therefore \boxed{\cos 2\phi = \cos^2\phi - \sin^2\phi} \quad (d)$$

$$\boxed{\sin 2\phi = 2\cos\phi\sin\phi} \quad (e)$$

We can go further. Use $\sin^2\phi = 1-\cos^2\phi$ in (d)

$$\therefore \cos 2\phi = \cos^2\phi - (1-\cos^2\phi)$$
$$= -1 + 2\cos^2\phi$$

$$\boxed{\therefore \cos^2\phi = \frac{1}{2}(1+\cos 2\phi)}$$ (f)

Use $\cos^2\phi = 1-\sin^2\phi$ in (d)

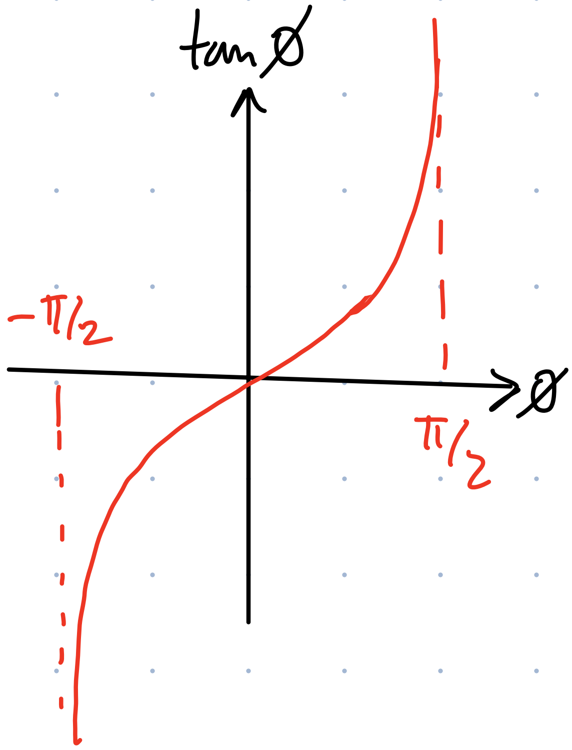
$$\therefore \cos 2\phi = (1-\sin^2\phi) - \sin^2\phi$$
$$= 1 - 2\sin^2\phi$$

$$\boxed{\therefore \sin^2\phi = \frac{1}{2}(1-\cos 2\phi)}$$ (g)

If $\pm j = \sqrt{-1}$, what is $\sqrt{\pm j} = (-1)^{1/4}$?

j is just a complex number $z = x + jy$

w/ $x = 0$ and $y = 1$



$$\therefore |z| = \sqrt{x^2 + y^2} = 1$$

$$\tan\phi = \frac{y}{x} \rightarrow \infty$$

$\tan\phi \rightarrow \pm\infty$ corresponds to

$$\phi = \pm \pi/2$$

$$\therefore \pm j = |z| e^{j\phi} = e^{\pm j\pi/2}$$

$$\therefore \sqrt{j} = \left(e^{\pm j\pi/2}\right)^{1/2} = e^{\pm j\pi/4}$$

$$= \underbrace{\cos\left(\frac{\pm\pi}{4}\right)}_{\frac{1}{\sqrt{2}}} + j\underbrace{\sin\left(\frac{\pm\pi}{4}\right)}_{\pm\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}}(1 \pm j)$$

Check: $\frac{1}{\sqrt{2}}(1 \pm j)\frac{1}{\sqrt{2}}(1 \pm j) = \frac{1}{2}(\underbrace{1 + j^2}_{0} \pm 2j) = \pm j$ ✓

This kind of process can be repeated to evaluate

$(j)^{\frac{1}{n}}$ for, say, any integer n.

What about j^j?

Consider Euler's eq'n: $e^{j\phi} = \cos\phi + j\sin\phi$

sub $\phi = \pi/2$, as above.

$$e^{j\pi/2} = j$$

but we can add any integer multiple of 2π to $\frac{\pi}{2}$ and get the same result.

$$\therefore\ j = e^{j(\frac{\pi}{2} + 2n\pi)} = e^{j\frac{\pi}{2}(1+4n)}$$
$$= e^{j\frac{\pi}{2}} e^{j2\pi n}$$

$\therefore$ if $z = j^j$ $\quad \ln z = j \ln j$

$$= j \ln \left[e^{j\frac{\pi}{2}} e^{j2\pi n} \right]$$

$$= j \left[\ln e^{j\pi/2} + \ln e^{j2\pi n} \right]$$

$$= j \left[j \frac{\pi}{2} + j 2\pi n \right]$$

$$= -\frac{\pi}{2} - 2\pi n$$

$$\therefore j^j = e^z = e^{-\pi/2 - 2\pi n} = e^{-\frac{\pi}{2}(1+4n)}$$

$\therefore j^j$ is a real <u>and</u> multivalued!

In general, for any complex no. $z = |z| e^{j\phi}$

$$= |z| e^{j(\phi + 2n\pi)}$$

$$\ln z = \ln \left[|z| e^{j(\phi + 2n\pi)} \right]$$

$$\boxed{\ln z = \ln |z| + j(\phi + 2n\pi)}$$

The ln of a complex no. is multivalued.

Euler's eq'n $e^{j\phi} = \cos\phi + j\sin\phi$ can be used to uncover what some call "the most beautiful equation in math". Sub in $\phi = \pi$

$$e^{j\pi} = -1 \Rightarrow e^{j\pi} + 1 = 0$$

This expression combines many of the famous constants from math: 0, 1 of the real number line, j from the imaginary no. line, and the transcendental nos. e & π.

However, the real beauty is in Euler's equation itself. $e^{j\pi}$ is just a special case.

Maybe, for some reason, I want an irrational number like $\sqrt{2}$ to appear in my equation. Try $\phi = \frac{\pi}{4}$. Then:

$$e^{j\pi/4} = \frac{1}{\sqrt{2}}(1+j)$$

One more... What if I wanted the Golden ratio φ to appear in my Euler-like identiy.

$$\varphi = 1 + \frac{1}{\varphi} \Rightarrow \varphi^2 - \varphi - 1 = 0$$

$$\therefore \varphi = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

φ can also be expressed as the following continued fraction

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

and is said to be the "most" irrational number.

Try $\phi = \frac{\pi}{5}$

$$e^{\pm j\pi/5} = \cos\frac{\pi}{5} \pm j \sin\frac{\pi}{5}$$

$$\frac{\varphi}{2} \rightsquigarrow (1+\sqrt{5})/4 \qquad \sqrt{5-\sqrt{5}}\,\sqrt{2}/4$$

$$\therefore e^{\pm j\pi/5} = \frac{\varphi}{2} \pm j\,\frac{\sqrt{5-\sqrt{5}}\,\sqrt{2}}{4}$$

$$\therefore e^{j\pi/5} + e^{-j\pi/5} = \varphi$$

or

$$\boxed{e^{j\pi/5} + e^{-j\pi/5} - \varphi = 0}$$

which is really just

$$\boxed{2\cos\left(\frac{\pi}{5}\right) - \varphi = 0}$$

Complex exponentials representing waves in physics...

As you move through your physics degree, you will encounter the wave eq'n in multiple different contexts. It is of the form:

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

One form of the single-freq/harmonic sol'n is

$$\Psi = \underbrace{A\cos(kx-\omega t) + B\sin(kx-\omega t)}_{\text{forward traveling wave}} \qquad (\#)$$

$$+ \underbrace{C\cos(kx+\omega t) + D\sin(kx+\omega t)}_{\text{backward traveling wave.}}$$

the ratio A/B (C/D) determines the phase of the forward (backward) wave.

You can confirm that $\frac{\partial^2 \Psi}{\partial x^2} = -k^2 \Psi$, $\frac{\partial^2 \Psi}{\partial t^2} = -\omega^2 \Psi$

$$\therefore \cancel{-} k^2 \cancel{\Psi} = \frac{1}{v^2}(\cancel{-}\omega^2 \cancel{\Psi})$$

$v = \frac{\omega}{k}$ is the wave speed.

Recall that we were able to write

$$\left.\begin{aligned} \cos\phi &= \frac{e^{j\phi}+e^{-j\phi}}{2} \\ \sin\phi &= \frac{e^{j\phi}-e^{-j\phi}}{2j} \end{aligned}\right\} \text{sub into } \Psi.$$

$$\therefore \Psi(x,t) = \frac{A}{2}\left(e^{j(kx-\omega t)} + e^{-j(kx-\omega t)}\right) + \frac{B}{2j}\left(e^{j(kx-\omega t)} - e^{-j(kx-\omega t)}\right)$$

$$+ \frac{C}{2}\left(e^{j(kx+\omega t)} + e^{-j(kx+\omega t)}\right) + \frac{D}{2j}\left(e^{j(kx+\omega t)} - e^{-j(kx+\omega t)}\right)$$

$$= \underbrace{\left(\frac{A-jB}{2}\right)}_{\equiv \tilde{A}} e^{j(kx-\omega t)} + \underbrace{\left(\frac{A+jB}{2}\right)}_{\equiv \tilde{B}} e^{-j(kx-\omega t)}$$

$$+ \underbrace{\left(\frac{C-jD}{2}\right)}_{\equiv \tilde{C}} e^{j(kx+\omega t)} + \underbrace{\left(\frac{C+jD}{2}\right)}_{\equiv \tilde{D}} e^{-j(kx+\omega t)}$$

$$\therefore \Psi(x,t) = \tilde{A}e^{j(kx-\omega t)} + \tilde{B}e^{-j(kx-\omega t)} + \tilde{C}e^{j(kx+\omega t)} + \tilde{D}e^{-j(kx+\omega t)} \quad \text{④}$$

$\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ are complex coeffients.

If # is a sol'n to the wave equation w/ $v = \frac{\omega}{k}$,
then so is ⊛.

⊛ is often a much more convenient form of the solution for two major reasons.

$\boxed{\#1}$ We can separate the time and position dependence

$$e^{j(kx-\omega t)} = e^{jkx} e^{-j\omega t}$$

$$e^{-j(kx-\omega t)} = e^{-jkx} e^{j\omega t}$$

$$\vdots$$

⊛ becomes

$$\Psi = \tilde{A} e^{jkx} e^{-j\omega t} + \tilde{B} e^{-jkx} e^{j\omega t} + \tilde{C} e^{jkx} e^{j\omega t} + \tilde{D} e^{-jkx} e^{-j\omega t}$$

$$= \underbrace{e^{-j\omega t}\left(\tilde{A} e^{jkx} + \tilde{D} e^{-jkx}\right)}_{\Psi_-} + \underbrace{e^{j\omega t}\left(\tilde{C} e^{jkx} + \tilde{B} e^{-jkx}\right)}_{\Psi_+}$$

Examine the real part of Ψ_-

$$Re[\Psi_-] = Re\left[\tilde{A} e^{j(kx-\omega t)} + \tilde{D} e^{-j(kx+\omega t)}\right]$$

$$Re\left\{\left(\frac{A-jB}{2}\right)\left[\cos(kx-\omega t) + j\sin(kx-\omega t)\right]\right.$$

$$\left. + \left(\frac{C-jD}{2}\right)\left[\cos(kx+\omega t) + j\sin(kx+\omega t)\right]\right\}$$

$$= \frac{A}{2}\cos(kx-\omega t) + \frac{B}{2}\sin(kx-\omega t)$$

$$+ \frac{C}{2}\cos(kx+\omega t) + \frac{D}{2}\sin(kx+\omega t)$$

$$= \frac{1}{2}\Psi.$$

$\therefore$ $Re[\Psi_-]$ contains forwards and backwards waves and all of the original constants A, B, C, D.

In practice, when faced w/ the wave equation:

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2},$$

physicists usually assume a sol'n of the form

$$\Psi(x,t) = \psi(x)\,\phi(t) \Rightarrow \text{separation of variables}$$

where $\psi(x) = \tilde{Q}\, e^{jkx} + \tilde{R}\, e^{-jkx}$

and $\phi(t) = e^{-j\omega t}$

$\therefore$ the x & t dependencies have been separated into two fcns that can be analyzed separately. Here, $\tilde{Q}$ and $\tilde{R}$ are coefficients independent of A, B, C, D and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$.

The boundary conditions determine $\tilde{Q}$ & $\tilde{R}$. If we want, we can take the $\mathrm{Re}[\Psi]$ at the end to recover a real sol'n. However, in practice, this is rarely done.

#2 The second advantage is that derivates and integrals become simple products when using using complex exponentials.

$$\text{Suppose } f(x,t) = Ae^{j(kx-\omega t)}$$

$$\frac{\partial f}{\partial x} = jk f(x,t) \quad ; \quad \frac{\partial f}{\partial t} = -j\omega f(x,t)$$

$$\int f\,dx = \frac{1}{jk} f(x,t) \; ; \; \int f\,dx = -\frac{1}{j\omega} f(x,t)$$

Trig fcns, on the other hand change $\cos \Leftrightarrow \sin$ after derivatives and integrals. Mixtures of sines and cosines, while manageable, are mathematically awkward/inconvenient.

Return to $v = \frac{\omega}{k}$.

Suppose Ψ represents light traveling through a medium. In this case

$v = \frac{c}{n}$ where n is the index of refraction.

In real materials, in addition to refracting light, also absorb some of the light as it passes through. This effect is modeled by adding an "extinction coefficient" K to n:

$$n + jK, \text{ where } K \ll n \text{ \& } K > 0$$

Let's examine why it's added w/ a $-j$ factor included

$$\therefore v = \frac{c}{n+jK} = \frac{\omega}{k} \qquad \therefore k = \frac{\omega}{c}(n+jK)$$

$$= k' + jk''$$

Sub into $\psi(x) = \tilde{Q} e^{jkx} + \tilde{R} e^{-jkx}$

$$= \tilde{Q} e^{j(k'+jk'')x} + \tilde{R} e^{-j(k'+jk'')x}$$

$$= \underbrace{\tilde{Q} e^{-k''x} e^{jk'x}}_{} + \underbrace{\tilde{R} e^{k''x} e^{-jk'x}}_{}$$

forward traveling wave is attenuated by the $e^{-k''x}$ factor as x increases

backward traveling wave is attenuated by the $e^{k''x}$ factor as x <u>decreases</u>. i.e. $\propto e^{-k''|\Delta x|}$, where $\Delta x < 0$.

So here $k = k' + jk''$ is telling use that there is attenuation of the wave as it propagates through a lossy material.

Complex Numbers in Electronics.

The voltage across a resistor is very conveniently expressed as $v_R = iR$

We'd like similar relationships for inductors and capacitors.

$$v_L = i Z_L$$

$$v_C = i Z_C$$

where Z_L & Z_C are the impedance of inductors and capacitors, respectively.

We assume that the current is sinusoidal and of the form $i = I_0 \sin \omega t$, which we express as $i = I_0 e^{j\omega t}$.

By Faraday's Law, the voltage across an inductor is given by:

$$v_L = L\frac{di}{dt} = L\frac{d}{dt}\left(I_0 e^{j\omega t}\right) = \underbrace{j\omega L}_{Z_L}\, i$$

$$\boxed{\therefore\ v_L = i Z_L \ , \text{ where } Z_L = j\omega L}$$

harmonic signals only.

For a capacitor, $C = \frac{q}{v_c}$ $\therefore$ $i = \frac{dq}{dt} = C\frac{dv_c}{dt}$

$$\therefore\ I_0 e^{j\omega t} = C\frac{dv_c}{dt}$$

$$\Downarrow$$

$$\frac{dv_c}{dt} = \frac{1}{C} I_0 e^{j\omega t}$$

$$\Downarrow$$

$$v_c = \frac{1}{C}\int I_0 e^{j\omega t}\,dt$$

$$\Downarrow$$

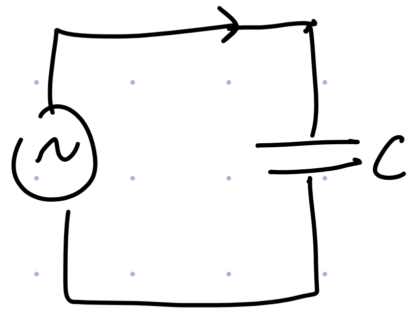
$$v_c = \underbrace{\frac{1}{j\omega C}}_{Z_c}\, i$$

$$\boxed{\therefore\ v_c = i Z_c, \text{ where } Z_c = \frac{1}{j\omega C}}$$

harmonic signals only

Impedances combine in series and parallel just like resistors. How do we interpret the j and $\frac{1}{j}$ in Z_L and Z_C?

Start w/ the following simple circuit



$$v_C = i Z_C$$
$$= \frac{1}{j\omega C} I_0 e^{j\omega t}$$

Recall that $j = e^{j\pi/2}$

$$\therefore v_C = \left(\frac{I_0}{\omega C}\right) \frac{e^{j\omega t}}{e^{j\pi/2}} = \underbrace{\left(\frac{I_0}{\omega C}\right)}_{\text{amplitude of the voltage across the cap}} e^{j(\omega t - \underbrace{\pi/2}_{\text{phase of the cap voltage.}})}$$

$$\boxed{\therefore v_C = V_C e^{j(\omega t - \phi)} \qquad V_C = \frac{I_0}{\omega C} \qquad \phi = \pi/2}$$

We could have obtained this result in another way:

$$Z_c = \frac{1}{j\omega C}\frac{j}{j}$$ mult. by one.

$$= \frac{-j}{\omega C}$$ use $-j = e^{-j\pi/2}$

$$= \frac{e^{-j\pi/2}}{\omega C}$$

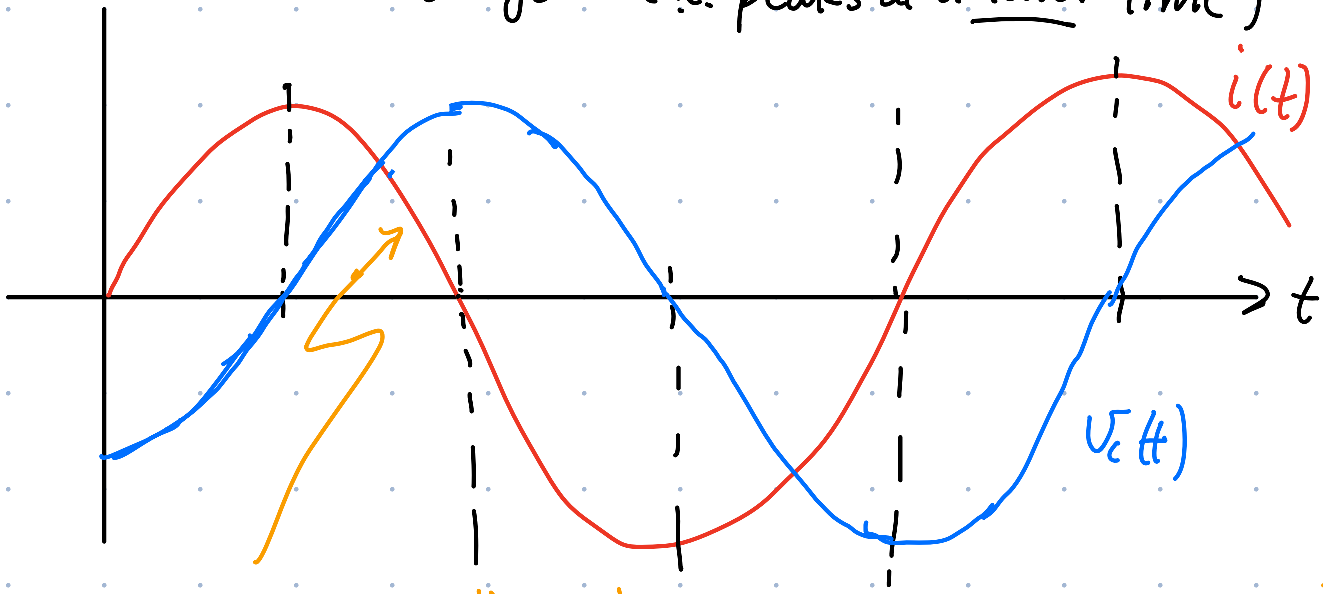
$$\therefore\ v_c = i Z_c = \left(I_0 e^{j\omega t}\right)\left(\frac{e^{-j\pi/2}}{\omega C}\right)$$

$$= \left(\frac{I_0}{\omega C}\right)e^{j(\omega t - \pi/2)}$$ same as before!

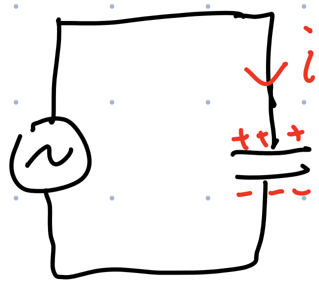
The j is Z_c is encoding the phase difference between the cap current and voltage.

v_c $\boxed{\text{lags}}$ i by $\pi/2$

v_c lags i (ie. peaks at a later time)

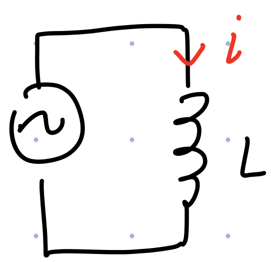


here i is still positive (although past its peak). A positive current continues to deposit positive charge on top plate of the cap.



If i is pos. but decreasing, the cap voltage is still increasing.

We can repeat this type of analysis for an inductor.



$$Z_L = j\omega L \qquad j = e^{j\pi/2}$$

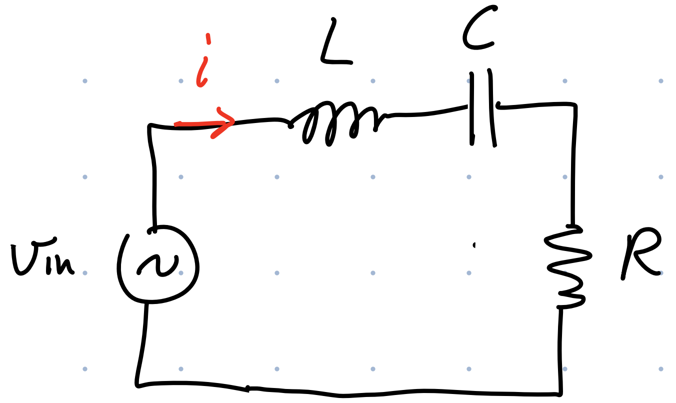
$$\therefore Z_L = \omega L e^{j\pi/2}$$

$$\therefore v_L = i Z_L = \left(I_0 e^{j\omega t}\right)\left(\omega L e^{j\pi/2}\right)$$

$$\boxed{\begin{aligned} \therefore\ v_L &= V_L e^{j(\omega t - \phi)} = (\omega L I_0) e^{j(\omega t - (-\pi/2))} \\ \therefore\ V_L &= \omega L I_0 \\ \phi &= -\pi/2 \end{aligned}}$$

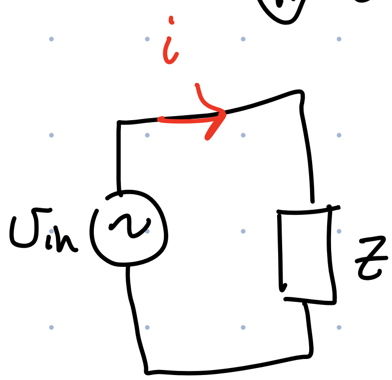
The inductor voltage <u>leads</u> the current by $\pi/2$.

Final example. LRC series circuit.



Calculate the current if $v_{in} = V_0 e^{j\omega t}$

$\Downarrow$ combine impedances



$$Z = R + Z_L + Z_C$$

$$= R + j\omega L + \frac{1}{j\omega C}\,\frac{j}{j}$$

$$\therefore\ Z = R + j\left(\omega L - \frac{1}{\omega C}\right) = x + jy$$

$$x = R \qquad y = \omega L - \frac{1}{\omega C}$$

$$\therefore z = |z| e^{j\phi_z} \quad \text{where:}$$

$$|z| = \sqrt{x^2 + y^2} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

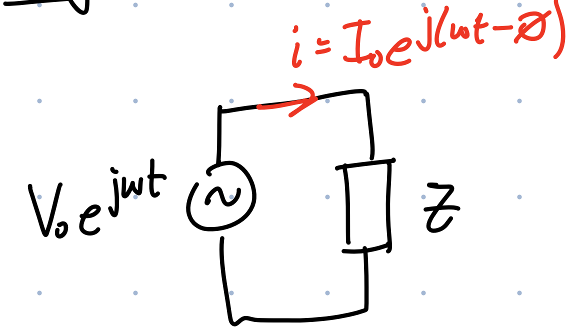
$$= R\sqrt{1 + \left(\omega \frac{L}{R} - \frac{1}{\omega R C}\right)^2}$$

$$\tan \phi_z = \frac{y}{x} = \frac{\omega L - \frac{1}{\omega C}}{R} = \omega \frac{L}{R} - \frac{1}{\omega R C}$$

Now $$i = \frac{V_{in}}{z} = \frac{V_0 e^{j\omega t}}{|z| e^{j\phi_z}}$$

$$\therefore i = I_0 e^{j(\omega t - \phi)} = \frac{V_0}{|z|} e^{j(\omega t - \phi_z)}$$

For <u>any</u> circuit of the form:

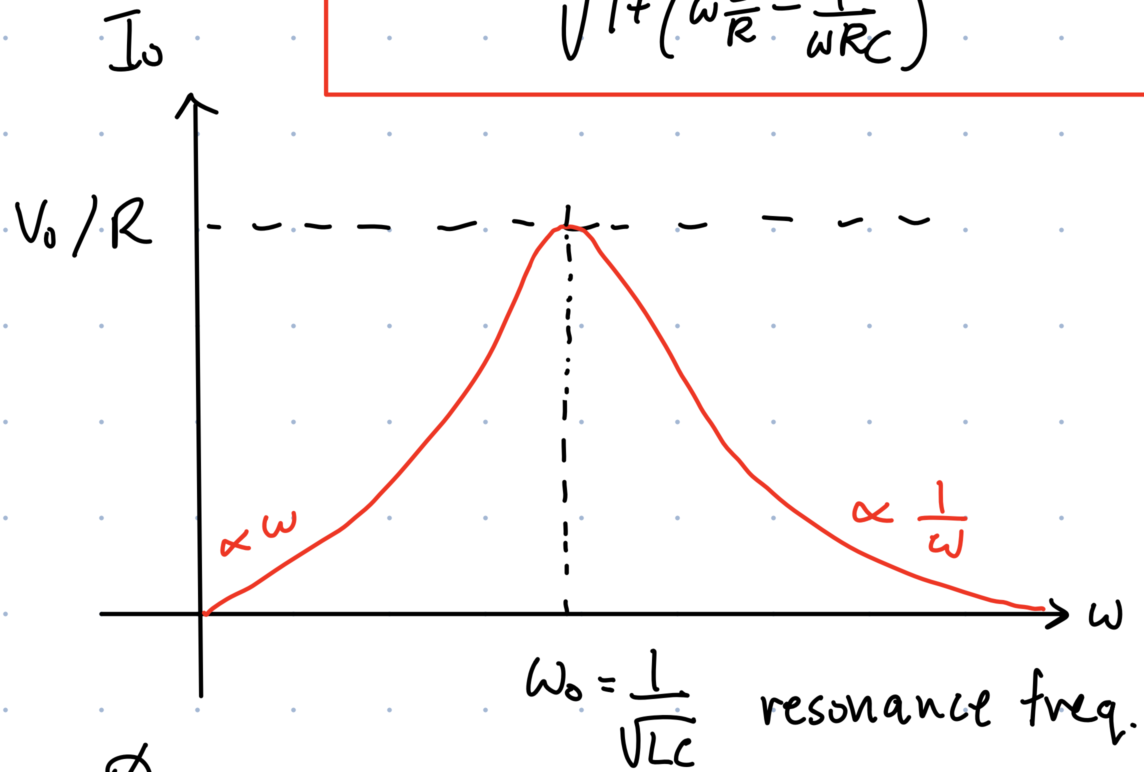


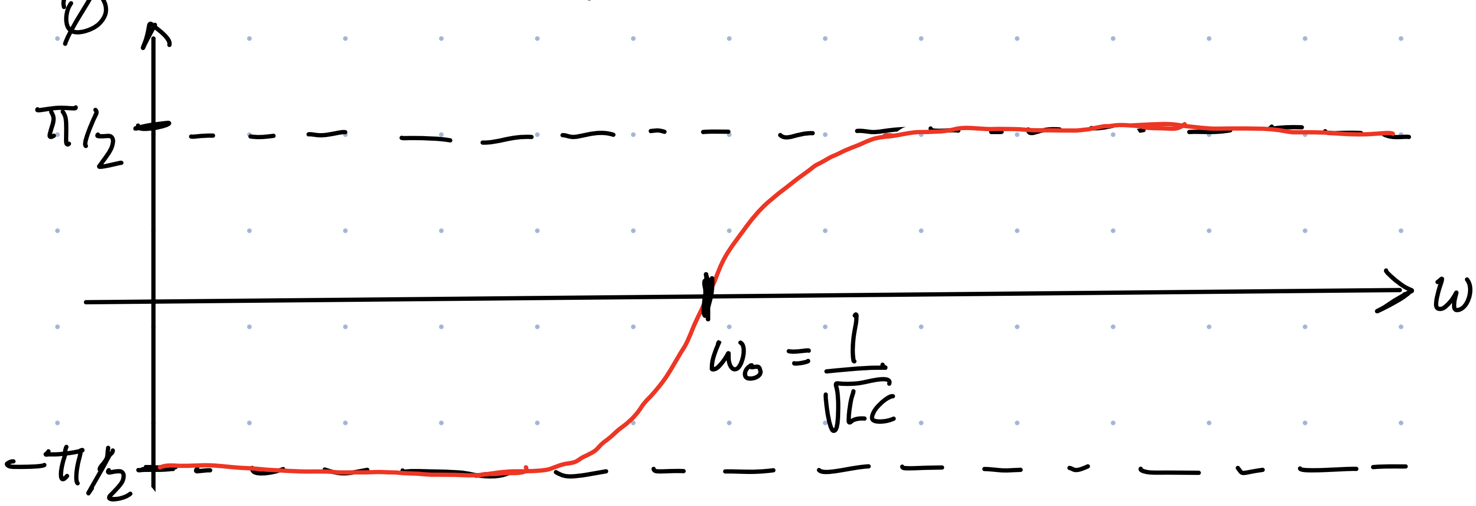
$$\boxed{I_0 = \frac{V_0}{|z|} \qquad \phi = \phi_z}$$

The magnitude $|Z|$ and phase ϕ_Z of the impedance Z determine the amplitude I_0 and phase ϕ of the current i.

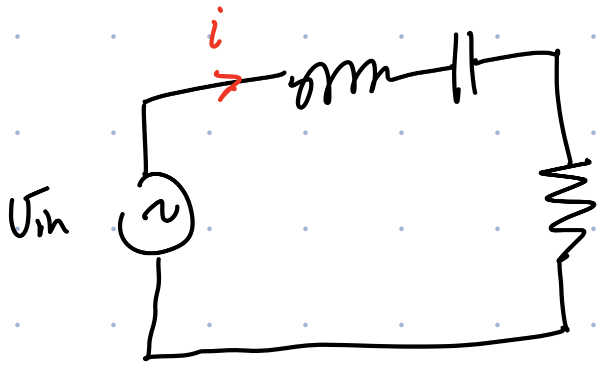
For the series LRC circuit:

$$\boxed{I_0 = \frac{V_0/R}{\sqrt{1+\left(\omega\frac{L}{R}-\frac{1}{\omega RC}\right)^2}} \qquad \tan\phi = \omega\frac{L}{R}-\frac{1}{\omega RC}}$$





Much easier than trying to start w/ the differential eq'n that you get from a Kirchhoff loop analysis.



$$V_{in} - V_L - V_C - V_R = 0$$

$$\therefore V_{in} - L\frac{di}{dt} - \frac{1}{C}q - iR = 0$$

$$\boxed{\therefore \frac{dV_{in}}{dt} = L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i}$$

Our complex impedances have allowed us to by-pass this 2nd order differential eq'n.

Although, using what we know about complex exponentials, we can make quick progress here too!

sub into diff. eq'n.

$$V_{in} = V_0 e^{j\omega t} \qquad \frac{dV_{in}}{dt} = j\omega V_0 e^{j\omega t}$$

$$i = I_0 e^{j(\omega t - \phi)} = I_0 e^{j\omega t} e^{-j\phi}$$

$$\frac{di}{dt} = j\omega I_0 e^{j\omega t} e^{-j\phi}$$

$$\frac{d^2 i}{dt^2} = -\omega^2 I_0 e^{j\omega t} e^{-j\phi}$$

$$j\omega V_0 \cancel{e^{j\omega t}} = -\omega^2 L I_0 \cancel{e^{j\omega t}} e^{-j\phi} + j\omega R I_0 \cancel{e^{j\omega t}} e^{-j\phi} + \frac{I_0}{C} \cancel{e^{j\omega t}} e^{-j\phi}$$

$$\therefore j\omega V_0 = I_0 e^{-j\phi} \left(-\omega^2 L + j\omega R + \frac{1}{C} \right)$$

$$\therefore \cancel{j\omega} V_0 = I_0 e^{-j\phi} (\cancel{j\omega}) \left(R + j\omega L + \frac{1}{j\omega C} \right)$$

$$\therefore I_0 e^{-j\phi} = \frac{V_0}{R + j\omega L + \frac{1}{j\omega C}}$$

$Z = R + Z_L + Z_C$

equivalent to the analysis we completed above!